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MULTIPOLE POTENTIALS FOR $SU(n)$ and $SO(n)$

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1. It is of physical interest to understand the symmetry properties of combinations of objects with known symmetries. A well-known elementary example is the Maxwell-Sylvester analysis of electric potentials generated by finite configurations of point charges [1]. In Euclidean $\mathbf{R}^3$ with coordinates x, y, z and $r = (x^2 + y^2 + z^2)^{1/2}$ one can choose units so that the potential from a single point charge at the origin is $1/r$. For a dipole centered at 0 with axis of length $\underline{a}$ aligned along the x-axis, the potential at large distances from the origin is well-approximated by $a(\partial/\partial x)(1/r)$. Similarly a configuration of 4 charges

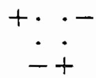

FIG. 1

as in Fig. 1 generates a quadrupole potential which is a multiple of $\partial^2/\partial y \partial x(1/r)$. In general, associated to any polynomial p there is a multipole potential $M(p) = \partial_p(1/r)$ where ∂_p is the constant coefficient differential operator corresponding to p via the Euclidean metric. Since $1/r$ is harmonic, away from its singularity at 0, $M(p) = 0$ if p is a multiple of r^2, and $M(p)$ is always harmonic. In addition, every polynomial may be written as $p = h + qr^2$ with h harmonic [6] so that M may be viewed as a mapping (hereafter referred to as the multipole mapping) from harmonic polynomials to singular harmonic functions.

In order to study objects with more complicated symmetry than the spherically symmetric point charge, one can replace Euclidean $\mathbf{R}^3$ with the Lie algebra $\mathfrak{g}$ of a compact simple Lie group G. Then G acts on $\mathfrak{g}$ via the adjoint representation, preserving the positive definite metric B, where $-B$ is the Killing form. Orbits of maximal dimension will be said to be regular; all other orbits will be said to be singular. There is an invariant polynomial Q on $\mathfrak{g}$ (for $\mathfrak{su}(n)$, Q is just the discriminant of the characteristic polynomial) which

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vanishes precisely on the singular set, and such that $1/\sqrt{Q}$ is harmonic on the regular set [3]. Here harmonic means annihilated by all G-invariant constant coefficient differential operators. Thus there is a multipole mapping M: {harmonic polynomials} $\to$ {harmonic functions on the regular set} given, as before, by $M(p) = \partial_p(1/\sqrt{Q})$.

The main result of this paper is that, at least for $\mathfrak{g} = \mathfrak{su}(n)$ or $\mathfrak{so}(n)$, the map M is injective. (It is obvious that M is injective as a map to distributions on $\mathfrak{g}$. The point is that no distribution $\partial_p(1/\sqrt{Q})$ is supported on the singular set.)

§2 gives the proof of injectivity. In §3 this result is reinterpreted in terms of the nonvanishing of the term of lowest homogeneity in the asymptotic expansion of an oscillatory integral.

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Finally, in light of the recent work of A. Koranyi [5] it may be worth pointing out that the multipole $M(p)$ and the Kelvin transformation Kp agree in case p is a homogeneous harmonic polynomial on $\mathbf{R}^3 = \mathfrak{su}(2)$, but in no other cases. The Kelvin transformation is the transformation of harmonic functions on $\mathbf{R}^n$ given by

$$Kf(X) = \frac{1}{\|x\|^{n-2}} f\left(\frac{x}{\|x\|^2}\right).$$

2. Lemma. *Let $a_1, \cdots, a_k$ be positive integers. Let $L_1, \cdots, L_k$ be distinct linear (affine) functions on $\mathbf{R}^N$ (or $\mathbf{C}^N$), and let $\partial_1, \cdots, \partial_s$ be constant vector fields on $\mathbf{R}^N$ (or $\mathbf{C}^N$) such that for each i there exists j with $\partial_i(L_j) \neq 0$. Then*

$$\partial_s \cdots \partial_1\left((L_1^{a_1} \cdots L_k^{a_k})^{-1}\right) \not\equiv 0.$$

Proof. By induction on N. The case $N = 1$ follows from the partial fractions decomposition of $(L_1^{a_1} \cdots L_k^{a_k})^{-1}$. For the induction, assume $\partial_1, \cdots, \partial_r$ satisfy $\partial_i(L_1) \neq 0$, and $\partial_{r+1}, \cdots, \partial_s$ satisfy $\partial_i(L_1) = 0$. Then the only term in $\partial_s \cdots \partial_1((L_1^{a_1} \cdots L_k^{a_k})^{-1})$ with a pole of order $r + a_1$ along $L = 0$ is

$$\begin{aligned}(\partial_r \cdots \partial_1)(L_1^{a_1})^{-1}(\partial_s \cdots \partial_{r+1})\left((L_2^{a_2} \cdots L_k^{a_k})^{-1}\right) \\ = c(L_1)^{-(r+a_1)}(\partial_s \cdots \partial_{r+1})\left((L_2^{a_2} \cdots L_k^{a_k})^{-1}\right)\end{aligned}$$

with $c \neq 0$. Restricted to any level set of L_1, which has dimension $n - 1$, $(\partial_s \cdots \partial_{r+1})((L_2^{a_2} \cdots L_k^{a_k})^{-1})$ is a nonzero rational function.

Let $\mathfrak{g}$ be the Lie algebra of a compact simple Lie group G. Let $G_{\mathbf{C}}$ be the corresponding complex group with Lie algebra $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a positive system of roots Φ for $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{h}$. Let W be the Weyl group. Throughout G acts on $\mathfrak{g}$ by the adjoint representation.

If p is a G-invariant polynomial on $\mathfrak{g}$, let $\partial(p)$ be the corresponding constant coefficient differential operator on $\mathfrak{g}$ (duality is established by the Killing form). Similarly for a polynomial q on $\mathfrak{h}$, $\partial(q)$ indicates the corresponding differential operator on $\mathfrak{h}$. Let $\Pi: \mathfrak{h} \to \mathbf{C}$ be defined by $\Pi(H) = \Pi_{\alpha \in \Phi} \alpha(H)$.

Theorem 1 (*Harish-Chandra*). *Let f be a G-invariant function on $\mathfrak{g}$. Let $H \in \mathfrak{h}$ be an element of a regular orbit, and p a G-invariant polynomial on $\mathfrak{g}$. Then*

$$(\partial(p)f)(H) = \frac{1}{\Pi(H)} \cdot [\partial(p|_{\mathfrak{h}})(\Pi \cdot f|_{\mathfrak{h}})](H). \tag{1}$$

Sketch of proof (see also [3], [4]). (1) may be established for the Laplace operator using the following three facts:

(i) Laplacian = divergence of gradient.

(ii) For invariant functions, gradient in $\mathfrak{g}$ and gradient in $\mathfrak{h}$ coincide.

(iii) Divergence is the Lie derivative of volume, and the $(n-l)$-dimensional volume of the G-orbit through H is $[\Pi(H)]^2$ up to a constant. Here $n = \dim \mathfrak{g}$ and $l = \dim \mathfrak{h}$.

It now follows from Vergne's argument [9] that $\mathcal{F}(f)|_{\mathfrak{h}} = 1/\Pi \mathcal{F}_{\mathfrak{h}}(\Pi f|_{\mathfrak{h}})$, where $\mathcal{F}_X$ is Fourier transform in the Euclidean space X. Therefore (1) is valid for all invariant polynomials p.

Let $\Pi^2 = \Pi_{\alpha \in \Phi}[\alpha(H)]^2$. Then Π^2 is a W-invariant polynomial on $\mathfrak{h}$ and therefore extends to a G-invariant polynomial Q on $\mathfrak{g}$.

Corollary. *The function $1/\sqrt{Q}$ is harmonic on the regular set of $\mathfrak{g}$.*

Proof. If all roots are multiplied by i (to make them real), Q will be positive on the entire regular set of $\mathfrak{g}$. Choose $\sqrt{Q}$ to be the positive square root, so that $1/\sqrt{Q}$ is an invariant function on the regular set of $\mathfrak{g}$. Now $\Pi \cdot 1/\sqrt{Q}\,|_{\mathfrak{h}}$ is locally constant on the regular set of $\mathfrak{h}$, so that (1) implies that $1/\sqrt{Q}$ is harmonic. Of course the corollary will also hold for any locally consistent choice of the square root.

Theorem 2. *Let $\mathfrak{g}$ be either $\mathfrak{su}(n)$ or $\mathfrak{so}(n)$, and let p be a harmonic polynomial on $\mathfrak{g}$. Then $\partial(p)(1/\sqrt{Q}) \not\equiv 0$ on the regular set.*

Proof. Let $\mathcal{Q}$ be the set of polynomials q on $\mathfrak{g}$ for which $\partial(q)(1/\sqrt{Q}) \equiv 0$ on the regular set. Then $\mathcal{Q}$ is an ideal and $\mathcal{Q} \supset \mathcal{I}$ where $\mathcal{I}$ is the ideal generated by the nonconstant invariant polynomials. Consider the corresponding varieties in $\mathfrak{g}_{\mathbf{C}}$. The variety $\mathcal{V}_{\mathcal{I}}$ corresponding to $\mathcal{I}$ is the nilpotent cone (clearly, if all invariant polynomials vanish, then all eigenvalues must vanish), so the variety

$\mathcal{V}_{\mathcal{Q}}$ must be contained in the nilpotent cone. Since the full polynomial ring $\mathcal{P}$ may be decomposed as $\mathcal{P} = \mathcal{H} \oplus \mathcal{I}$, if $\mathcal{Q} \cap \mathcal{H} \neq 0$ then $\mathcal{V}_{\mathcal{Q}}$ must be properly contained in $\mathcal{V}_{\mathcal{I}}$. Also since $\mathcal{Q}$ is $G_{\mathbf{C}}$-invariant, $\mathcal{V}_{\mathcal{Q}}$ must be a union of $G_{\mathbf{C}}$-orbits, i.e., a proper $G_{\mathbf{C}}$-invariant subset of the nilpotents.

The nilpotent cone is made up of one open $G_{\mathbf{C}}$-orbit of regular elements, and the remaining orbits have positive codimension. For the classical algebras in their standard representations it is easy to check using [7] that the singular nilpotents are those which have rank less than the regular nilpotents. In all cases, if the standard representation has dimension n, the singular nilpotents have rank $\leqslant n-2$. Thus if $\mathcal{Q} \cap \mathcal{H} \neq 0$, the Nullstellensatz implies that $\mathcal{Q}$ contains a power of every $(n-1) \times (n-1)$ minor. Assume this is the case.

In the cases $\mathfrak{g} = \mathfrak{su}(n)$ or $\mathfrak{so}(n)$, the rank $-\ n$ algebra $\mathfrak{g}$ contains an essentially rank $-\ (n-1)$ subalgebra $\mathfrak{k}$ which contains regular elements of $\mathfrak{g}$. Specifically, $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \times \mathfrak{u}(n-1)) \simeq \mathfrak{u}(n-1)$ or $\mathfrak{so}(n-1)$. Let $K \subset G$ be the corresponding subgroup. Corresponding to the choice of $\mathfrak{k}$ is a K-invariant $(n-1) \times (n-1)$ minor of $\mathfrak{g}$ denoted by p. The K-invariant operator $\partial(p)$ when restricted to $\mathfrak{k}$ contains only derivatives tangent to $\mathfrak{k}$. Choose a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{k}$ which is contained in the Cartan subalgebra of $\mathfrak{g}$. Choose positive systems $\Phi_{\mathfrak{k}} \subset \Phi_{\mathfrak{g}}$. Let $\Pi = \Pi_{\alpha \in \Phi_{\mathfrak{g}}} \alpha$ and $\Pi_{\mathfrak{k}} = \Pi_{\alpha \in \Phi_{\mathfrak{k}}} \alpha$. Let $Q_{\mathfrak{g}}$ be the invariant polynomial on $\mathfrak{g}$ which restricts to $(\Pi_{\mathfrak{g}})^2$ on the Cartan subalgebra. Since $1/\sqrt{Q_{\mathfrak{g}}}$ restricts to a K-invariant function on $\mathfrak{k}$, Theorem 1 may be applied to the calculation of the restriction to $\mathfrak{k}$ of $\partial(p^m)(1/\sqrt{Q_{\mathfrak{g}}})$. (It is easy to check that Theorem 1 applies to the reductive algebra $\mathfrak{u}(n-1)$ as well as to the semisimple algebra $\mathfrak{su}(n-1)$.) But $p^m|_{\mathfrak{h}}$ is a product of linear functions and $\Pi_{\mathfrak{k}} \cdot (1/\sqrt{Q_{\mathfrak{g}}}|_{\mathfrak{h}}) = \Pi_{\mathfrak{k}} \cdot 1/\Pi_{\mathfrak{g}}$ is of the form $(L_1^{a_1} \cdots L_k^{a_k})^{-1}$ for some linear functions L_i and positive integers a_i. Apply the lemma to conclude that $\partial(p^m)(1/\sqrt{Q_{\mathfrak{g}}}) \not\equiv 0$ on the regular set of $\mathfrak{g}$, and hence that $\mathcal{Q} \cap \mathcal{H} = 0$.

3. Let p be a harmonic polynomial on $\mathfrak{g}$, and let $\mathcal{O} \subset \mathfrak{g}$ be a regular orbit. Define the Fourier transform of the restriction $p|_{\mathcal{O}}$ by

$$\mathcal{F}(p|_{\mathcal{O}})(X) = \int_{\mathcal{O}} e^{i\langle X, Y\rangle} p(Y)\, d\mu(Y), \tag{2}$$

where $d\mu$ is a G-invariant measure on $\mathcal{O}$.

Replace X by λX where λ is a large parameter. The resulting oscillatory integral has an asymptotic expansion in λ. It follows from Harish-Chandra's formula for the orbital integral that the expansion has only a finite number of terms. The term of lowest degree in λ is just $M(p)$. A few details follow.

Theorem 3 (*Harish-Chandra*). *Let H, H' be regular elements of $\mathfrak{h}$, and $\mathcal{O}$ the orbit of H'. Then there is a constant c so that*

$$\mathcal{F}(1|_{\mathcal{O}})(H) = \frac{c}{\Pi(H)\Pi(H')} \sum_{s\in W} \operatorname{sgn} s\, e^{i(sH, H')}. \tag{3}$$

Sketch of proof. Since $\mathcal{F}(1|_{\mathcal{O}})$ is a G-invariant eigenfunction for all the invariant differential operators on $\mathfrak{g}$, Theorem 1 implies that as a function of H, $\mathcal{F}(1|_{\mathcal{O}}) = q/\Pi(H)\cdot\sum_{s\in W}\operatorname{sgn} s\, e^{i(sH,H')}$ for some constant q.

A slightly different perspective on Theorem 3 is obtained by attempting to evaluate the integral (2) by the method of stationary phase.

Theorem 4 (*Stationary phase*).

$$\int_M e^{i\lambda\phi(x)}\alpha(x)\,dx \sim \sum_{\substack{y\in M\\ d\phi(y)=0}} \left(\frac{2\Pi}{\lambda}\right)^{\frac{1}{2}\dim M} \left|\det H_\phi(y)\right|^{-\frac{1}{2}} \cdot e^{\frac{1}{4}\Pi i \operatorname{sgn} H_\phi(y)}\, e^{i\lambda\phi(y)} \left(\sum_{k=0}^{\infty} \frac{1}{k!} R^k\alpha(y)\cdot\lambda^{-k}\right). \tag{4}$$

where $H_\phi(y)$ is the Hessian of ϕ at y, R is a 2nd order differential operator defined in terms of H_ϕ, and $\sim$ means asymptotic equality as $\lambda\to\infty$.

Proof. See [2].

When M is the orbit of H' and $\phi(g\cdot H') = (H, g\cdot H')$, the critical points of ϕ are exactly the points $s\cdot H'$ where $s\in W$. Replacing H by λH in (3) shows that, in this special case, only the leading term for each critical point in the expansion (4) is nonzero and that the asymptotic equality is an actual equality. Of course, one can obtain a formula for the integral (2) by applying the operator $\partial(p)$ to (3). Specifically, with $d = \#$ positive roots we have

$$\int_{\mathcal{O}} e^{i\lambda(H, g\cdot H')}\,d\mu(g\cdot H') = \frac{c}{\lambda^d\Pi(H)\Pi(H')} \sum_{s\in W} \operatorname{sgn} s\, e^{i\lambda(H,sH')}.$$

If $m = \deg p$, then applying $\partial(p)$ to the H variable on both sides gives

$$(i\lambda)^m \int_{\mathcal{O}} e^{i\lambda(H,g\cdot H')} p(g\cdot H')\,d\mu(g\cdot H') = \frac{c}{\lambda^d\Pi(H')}\partial(p)\frac{1}{\Pi(H)} \sum_{s\in W} \operatorname{sgn} s\, e^{i\lambda(H,sH')}.$$

With p fixed, H can be chosen generically within its G-orbit so that every derivative involved in the operator $\partial(p)$ will give a nonzero result when applied to the function $L(H) = (H, sH')$. Then it is clear that the only term on the right of degree $-d$ in λ is

$$\frac{c}{\lambda^d\Pi(H')} M(p) \sum_{s\in W} \operatorname{sgn} s\, e^{i\lambda(H,sH')}.$$

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